

# Discrete quantitative nodal theorem

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In the theory of Riemann manifolds, two basic theorems connect the geometry of the manifold to the spectrum of the Laplace operator on the manifold: Courant's Nodal Theorem and Cheeger's Inequality. Both of these have analogues in graph theory. Analogues of Cheeger's Inequality for graphs have been proved by Alon and Milman [2], Alon [1], Dodziuk and Kendall [4], Jerrum and Sinclair [7] and others. Analogues of the Nodal Theorem have been proved by van der Holst [5] and van der Holst, Lovász and Schrijver [6], and generalized by Colin de Verdière [3]. In this note we prove a result that can be thought of as a common generalization of the Nodal Theorem and (one direction of) Cheeger's Inequality for graphs.

To motivate our work, let us cite (informally) some results on the spectrum of the Laplacian of a graph. It is a basic simple fact that the graph is connected if and only if the smallest eigenvalue (which is always 0) has multiplicity one. The theorem of Alon and Milman gives a quantitative version of this: If the second smallest eigenvalue is at least  $\varepsilon > 0$ , then the graph is an expander, where the expansion rate depends on  $\varepsilon$ . The Lemma of van der Holst implies that if  $x$  is an eigenvector of the Laplacian of a connected graph  $G$  belonging to the second smallest eigenvalue, and this eigenvalue has multiplicity one, then the positive and negative supports of  $x$  induce connected subgraphs. A consequence of our results is that if the second and third eigenvalues of the Laplacian are at least  $\varepsilon$  apart, then the subgraphs induced by the positive and negative supports of  $x$  are expanders (in a weighted sense).

Let  $(w_i : i \in V)$  be a weighting of the nodes of the graph  $G = (V, E)$  with nonnegative weights. For  $S \subseteq V$  with  $0 < w(S) < w(V)$ , define

$$w(S) = \sum_{i \in S} w_i \quad \text{and} \quad \Phi_w(S) = \Phi_{G,w}(S) = \frac{\sum_{i \in S, j \in V \setminus S} \sqrt{w_i w_j}}{\min(w(S), w(V \setminus S))}.$$

We say that  $G$  is a  $c$ -expander with respect to  $w$  ( $c > 0$ ), if  $\Phi_w(S) \geq c$  for every subset  $S \subseteq V$  with  $0 < w(S) < w(V)$ .

To generalize this notion to multiway cuts, it is perhaps easier to formulate the contrapositive. We say that  $(G, w)$  is  $(k, c)$ -partitionable, if  $V$  has a partition  $V = S_1 \cup \dots \cup S_k$  into sets with  $w(S_i) > 0$  such that

$$\Phi_w(S_i) < c \tag{1}$$

for all  $1 \leq i \leq k$ . If  $k > 2$ , then merging two classes in such a partition the new class still satisfies (1), and hence every  $(k, c)$ -partitionable weighted graph is  $(k-1, c)$ -partitionable as well.

For a vector  $x \in \mathbb{R}^V$ , we denote by  $\text{supp}_+(x) = \{i \in V : x_i > 0\}$  and  $\text{supp}_-(x) = \{i \in V : x_i < 0\}$  its positive and negative support, and let  $G_x^+$  and  $G_x^-$  denote the subgraphs of  $G$  induced by  $\text{supp}_+(x)$  and  $\text{supp}_-(x)$  respectively.

**Theorem 1** *Let  $G = (V, E)$  be a graph and let  $\lambda_1 = 0 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$  be the eigenvalues of its Laplacian  $L$ . Let  $y$  be an eigenvector belonging to  $\lambda_k$  ( $1 \leq k \leq n$ ), and let  $w_i = y_i^2$  and  $c = (\lambda_{k+1} - \lambda_k)/2$ . Suppose that the weighted graph  $(G_y^+, w)$  is  $(a, c)$ -partitionable, and  $(G_y^-, w)$  is  $(b, c)$ -partitionable. Then  $a + b \leq k$ .*

For  $k = 2$ , we get the corollary:

**Corollary 2** *If  $y$  is an eigenvector belonging to  $\lambda_2$ , then both  $G_y^+$  and  $G_y^-$  are  $(\lambda_3 - \lambda_2)/2$ -expanders with respect to the weights  $y_i^2$ .*

**Proof.** Let us write  $G^+ = G_y^+$ ,  $V^+ = \text{supp}_+(y)$  and  $\Phi^+(S) = \Phi_{G_y^+, w}(S)$  for  $S \subseteq V^+$ , and define  $G^-$ ,  $V^-$  and  $\Phi^-$  analogously. Let  $V^+ = V_1 \cup \dots \cup V_a$  be a partition with  $\Phi^+(V_i) < c$ , and let  $V^- = V_{a+1} \cup \dots \cup V_{a+b}$  be an analogous partition. Let us assume (by way of contradiction) that  $a + b \geq k + 1$ ; we may assume (by merging partition classes) that  $a + b = k + 1$ .

Let  $M = L - \lambda_k I$ , so that  $My = 0$ . Let  $y^i \in \mathbb{R}^V$  denote the vector obtained from  $y$  by replacing all entries in  $V \setminus V_i$  by 0. So  $y^i \geq 0$  for  $1 \leq i \leq a$ ,  $y^i \leq 0$  for  $a + 1 \leq i \leq a + b$ , and  $y = y^1 + \dots + y^{k+1}$ . Let  $z_i = \|y^i\|$ ,  $\hat{y}^i = (1/z_i)y^i$ , and  $z = (z_1, \dots, z_{k+1})^\top$ . Consider the  $(k + 1) \times (k + 1)$  matrix  $B$  defined by

$$B_{ij} = \langle \hat{y}^i, M\hat{y}^j \rangle,$$

and let  $\mu_1 \leq \dots \leq \mu_{k+1}$  be its eigenvalues.

Let us start with some elementary properties of  $B$ . We have  $\langle y^i, My^j \rangle \leq 0$  if  $i \neq j$  and  $1 \leq i, j \leq a$ , which implies that  $B_{ij} \leq 0$  in this case. Similarly  $B_{ij} \leq 0$  if  $i \neq j$  and  $a + 1 \leq i, j \leq a + b$ , and  $B_{ij} \geq 0$  if  $i \leq a < j$ , or the other way around. Furthermore, we have

$$(Bz)_i = \sum_{j=1}^{k+1} \langle \hat{y}^i, M\hat{y}^j \rangle z_j = \sum_{j=1}^{k+1} \langle \hat{y}^i, My^j \rangle = \langle \hat{y}^i, My \rangle = 0.$$

Since the vectors  $\hat{y}^i$  ( $i = 1, \dots, k + 1$ ) are mutually orthogonal unit vectors, the matrix  $B$  is a principal submatrix of  $M$  in an appropriate orthonormal basis. By the Interlacing Eigenvalues Theorem, we have

$$\lambda_i - \lambda_k \leq \mu_i \quad (i = 1, \dots, k + 1). \quad (2)$$

Let  $C$  be a symmetric  $(k + 1) \times (k + 1)$  matrix, given by

$$C_{ij} = \begin{cases} B_{ij} & \text{if } i \neq j \text{ and either } i, j \leq a \text{ or } i, j \geq a + 1, \\ - \sum_{\substack{r \leq a, r \neq i \\ a+b \\ r > a, r \neq j}} \frac{z_r}{z_i} B_{ir} & \text{if } i = j \leq a, \\ - \sum_{\substack{r > a, r \neq j \\ a+b \\ r \leq a, r \neq i}} \frac{z_r}{z_i} B_{ir} & \text{if } i = j \geq a + 1, \\ 0, & \text{otherwise.} \end{cases}$$

The diagonal entries are chosen so that  $Cz = 0$ , and they can be estimated as follows:

$$C_{ii} = -\frac{1}{z_i^2} \sum_{r \leq a, r \neq i} \langle y^i, My^r \rangle = -\frac{1}{z_i^2} \langle y^i, M(y^i - y_+) \rangle = \Phi^+(V_i) < c \quad (3)$$

if  $i \leq a$ , and we get the same bound in the case when  $i > a$ .

It is easy to check that the matrix  $C - B$  is given by

$$(C - B)_{ij} = \begin{cases} -B_{ij} & \text{if } i \leq a < j \text{ or } j \leq a < i, \\ \sum_{r=1}^a \frac{z_r}{z_i} B_{ir} & \text{if } i = j \geq a + 1, \\ \sum_{r=a+1}^{a+b} \frac{z_r}{z_i} B_{ir} & \text{if } i = j \leq a, \\ 0, & \text{otherwise.} \end{cases}$$

This matrix is positive semidefinite. Indeed, we have  $(C - B)z = Cz - Bz = 0$ . Let  $D = \text{diag}(z)$ , then the matrix  $D^{-1}(C - B)D^{-1}$  has nonnegative entries in the diagonal, nonpositive entries everywhere else, and every row-sum is 0. So this matrix is diagonally dominant, and hence positive semidefinite, which implies that  $C - B$  is positive semidefinite. It follows that  $\mu_{k+1}$ , the largest eigenvalue of  $B$ , is bounded above by the largest eigenvalue  $\lambda_{\max}(C)$  of the matrix  $C$ . Hence by (2),

$$\lambda_{k+1} - \lambda_k \leq \mu_{k+1} \leq \lambda_{\max}(C). \quad (4)$$

We claim that

$$\lambda_{\max}(C) < 2c. \quad (5)$$

Indeed, let  $u$  be the eigenvector of  $C$  belonging to  $\lambda_{\max}(C)$ . We may assume that  $u_1 = z_1 > 0$  and  $|u_i| \leq z_i$  for all  $i$ . Then, using (3),

$$\lambda_{\max}(C)u_1 = \sum_j C_{1j}u_j \leq C_{11}u_1 + \sum_{i>1} |C_{1i}|z_i = 2C_{11}u_1 < 2cu_1.$$

This proves (5). Thus we have

$$\lambda_{k+1} - \lambda_k \leq \mu_{k+1} < 2c,$$

which contradicts the choice of  $c$ .  $\square$

The assertion does not remain true without the weights. Consider two isomorphic  $D$ -regular expanders  $G_1$  and  $G_2$ , and connect them by a path on the same number of nodes. The gap  $\lambda_2 - \lambda_1 = \lambda_2$  will be small; the gap  $\lambda_3 - \lambda_2$  will be comparable with the eigenvalue gap of  $G_1$ . The eigenvector of  $\lambda_2$  will be close to a constant  $a$  on  $V(G_1)$ , a  $-a$  on  $V(G_2)$ , and small on the path. Its positive support will be  $V(G_1)$  together with half of the path; this is not an expander in the unweighted sense.

It follows by a similar argument that

**Proposition 3** *If  $y$  is an eigenvector belonging to  $\lambda_k$ ,  $a + b = k + 1$ , and  $\{V_1, \dots, V_a\}$  is a partition of  $V^+$  and  $\{U_1, \dots, U_b\}$  is a partition of  $V^-$ , then*

$$\lambda_{a+b} - \lambda_{a+b-1} \leq \sum_{i=1}^a \Phi^+(V_i) + \sum_{i=1}^b \Phi^-(U_i).$$

Indeed, notice that in the proof the matrix  $C$  is positive semidefinite, by a similar argument as for  $C - B$ . Hence its largest eigenvalue is bounded above by its trace. Since  $C - B$  is positive semidefinite, it follows that  $\mu_{a+b} \leq \text{tr}(C) = \sum_i C_{ii}$ . Substituting from (3), the inequality follows.

We remark in conclusion that it would be interesting to prove a converse theorem, stating that if a graph cannot be partitioned into more than  $k + 1$  weighted expanders, then there must be a large gap in the eigenvalue sequence.

## References

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